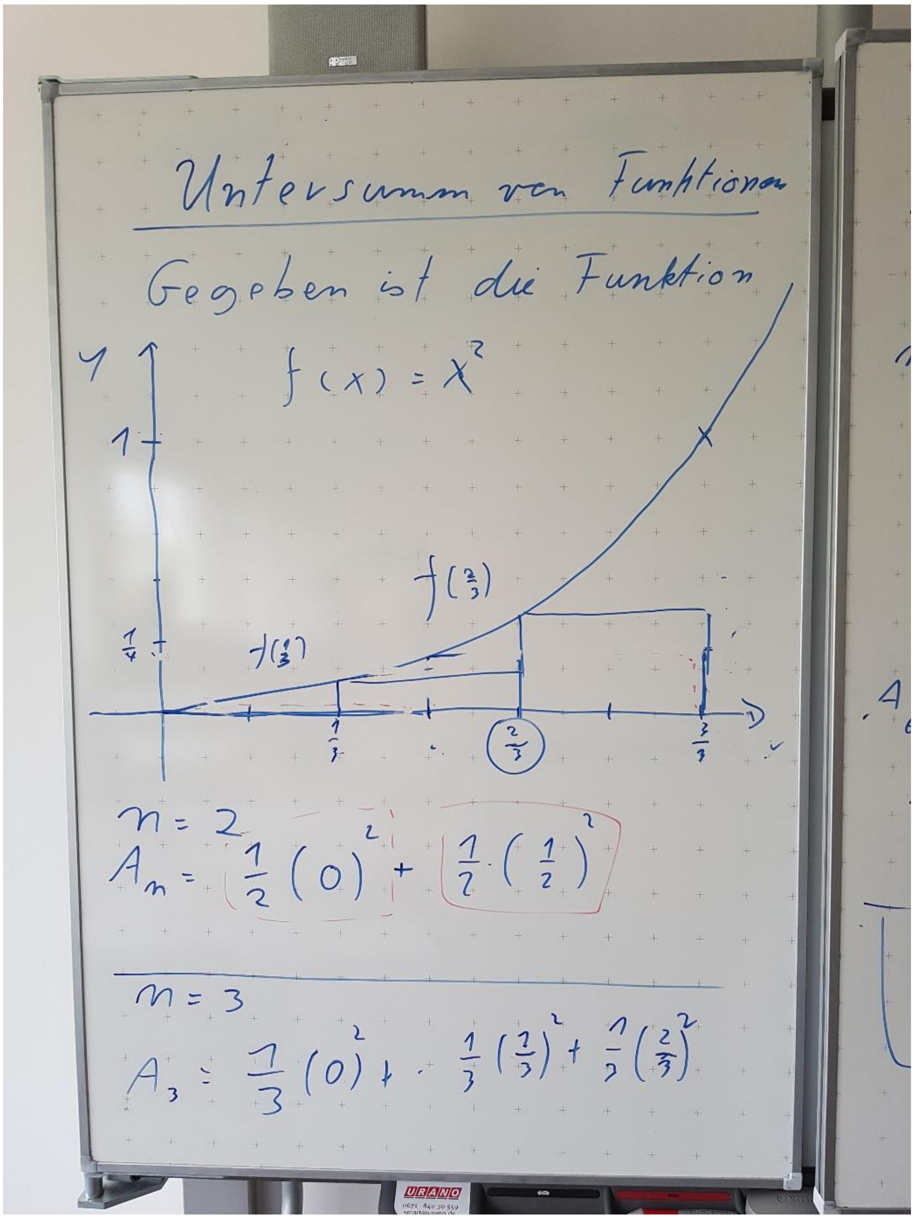
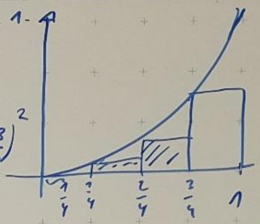
Von der Untersumme zum Flächeninhalt

$$A_n = \frac{1}{n} \cdot (0+1)^2 + \frac{1}{n} \cdot (\frac{1}{n}+1)^2 + \frac{1}{n} \cdot (\frac{2}{n}+1)^2 + \frac{1}{n} \cdot (\frac{3}{n}+1)^2 + \ldots + \frac{1}{n} \cdot (\frac{n-1}{n}+1)^2$$

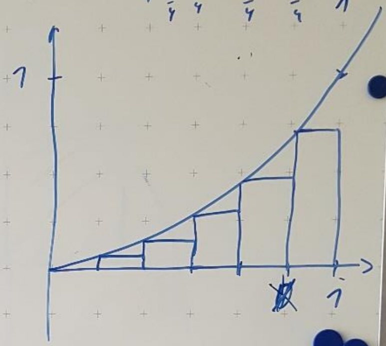


$n = 4$

$$A_4 = \frac{1}{4}(0)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{2}{4}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2$$



$n = 6$



$$A_6 = \frac{1}{6}(0)^2 + \frac{1}{6}\left(\frac{1}{6}\right)^2 + \frac{1}{6}\left(\frac{2}{6}\right)^2 + \frac{1}{6}\left(\frac{3}{6}\right)^2 +$$
$$\frac{1}{6}\left(\frac{4}{6}\right)^2 + \frac{1}{6}\left(\frac{5}{6}\right)^2$$

$$\boxed{\left(\frac{2}{n}\right)^2 = \frac{2^2}{n^2} = 2^2 \cdot \frac{1}{n^2} = 2^2 \cdot \left(\frac{1}{n}\right)^2}$$

,

$n = n$

$$A_n = \frac{1}{n}(0)^2 + \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \dots + \frac{1}{n}\left(\frac{n-1}{n}\right)^2$$

$$A_n = \frac{1}{n}\left[0^2 + \left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \dots + \left(\frac{n-1}{n}\right)^2\right]$$

$$A_n = \frac{1}{n}\left[0^2 + 1^2\left(\frac{1}{n}\right)^2 + 2^2\left(\frac{1}{n}\right)^2 + 3^2\left(\frac{1}{n}\right)^2 + \dots + (n-1)^2\cdot\left(\frac{1}{n}\right)^2\right]$$

$$A_n = \left(\frac{1}{n}\right)^3\left[0^2 + 1^2 + 2^2 + 3^2 + \dots + (n-1)^2\right]$$

$$A_n = \left(\frac{1}{n}\right)^3\left[1^2 + 2^2 + 3^2 + \dots + (n-1)^2\right]$$

Ohne Herleitung:

$$A_n = \left(\frac{1}{n}\right)^3 \cdot \frac{1}{6}\cdot(n-1)\cdot n\cdot(2n-1)$$

$$A_n = \frac{1}{n^3}\cdot\frac{1}{6}\cdot(n-1)\cdot n\cdot(2n-1)$$

$$A_n = \frac{1}{6} \cdot \frac{n-1}{n} \cdot \frac{n}{n} \cdot \frac{2n-1}{n}$$

$$A_n = \frac{1}{6} \cdot \frac{n}{n} - \frac{1}{n} \cdot 1 \cdot \frac{2n}{n} - \frac{1}{n}$$

$$A_n = \frac{1}{6} \left(1 - \frac{1}{n}\right) \cdot \left(2 - \frac{1}{n}\right)$$

$n \to \infty$

$$\lim_{n \to \infty} \frac{1}{6} \cdot \underbrace{\left(1 - \underbrace{\frac{1}{n}}_{\to 0}\right)}_{\to 1} \cdot \underbrace{\left(2 - \underbrace{\frac{1}{n}}_{\to 0}\right)}_{\to 2}$$

$$= \frac{1}{6} \cdot 1 \cdot 2 = \frac{2}{6} = \frac{1}{3} \text{ FE.}$$